A Root Locus Property of the Extended Bass Algorithm

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MANY aircraft stability and control problems may be formulated and solved through use of an nth order time-invariant vector equation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

with state vector x and control variable u. In the process of finding acceptable feedback control laws of the form

$$u(t) = Kx(t) \tag{2}$$

it is often required to find a constant gain matrix K such that the closed-loop response matrix $A+BK$ is asymptotically stable. That is, the eigenvalues of $A+BK$ lie strictly in the complex left-half plane. One easy-to-implement stabilization algorithm is stated in the following theorem.

Theorem 1 – Extended Bass Algorithm

Let $[A,B]$ be stabilizable. Then

$$K = -B'Z^{+} \tag{3}$$

stabilizes the system (1) where $Z = Z' \geq 0$ satisfies

$$-(A+\beta I)Z + Z[-(A+\beta I)]' = -2BB' \tag{4}$$

with $\beta > 0$ chosen such that the eigenvalues of $-(A+\beta I)$ are in the complex left-half-plane. The symbols $(\cdot)^{+}$ and $(\cdot)'$ denote matrix pseudoinverse and transpose, respectively.

A proof of Theorem 1 can be found in Ref. 1. The algorithm is an extension of a well-known method of Bass[2] for stabilizing single-input completely controllable systems to include multiple-input stabilizable systems. Shapiro and Decarli[3] present a simple method for determining β in terms of the elements of the $n \times n$ matrix A. The purpose of this Note is to detail a result concerning the root locus properties of the matrix $A+BK$ (for variations in β) which can also influence the choice of β in Eq. (4).

Theorem 2 – Root Locus Property

Let $[A,B]$ be stabilizable and $\beta > 0$ chosen such that the matrix $-(A+\beta I)$ is a stability matrix. Then the extended Bass algorithm (Theorem 1) yields a stabilizing gain K such that all eigenvalues λ of $A+BK$ that coincide with controllable eigenvalues of $A+BK$ satisfy

$$Re\lambda = -\beta \tag{5}$$

Proof

Let v be a left eigenvector of $A+BK$ with eigenvalue λ. That is,

$$(A+BK)'v = \lambda v \tag{6}$$

It follows from Ref. 1 that Eq. (4) can be written as

$$(A+BK)Z + Z(A+BK)' = -2\beta Z \tag{7}$$

Multiplying Eq. (7) on the right by v and on the left by v^*, where $(\cdot)^*$ denotes conjugate transpose, gives

$$(Re\lambda + \beta)v^*Zv = 0 \tag{8}$$

Equation (8) holds for each eigenvalue/left-eigenvector pair of $(A+BK)$. In order to distinguish between controllable and uncontrollable eigenvalues, partition the $(n \times 1)$ vector v as

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{9}$$

where v_1 is $\ell \times 1$, v_2 is $(n-\ell) \times 1$, and ℓ denotes the number of controllable modes of the $[A,B]$ pair. Additionally, from Ref. 1, there exists an orthogonal matrix T such that

$$\tilde{Z} = \begin{bmatrix} \tilde{Z}_{11} & 0 \\ 0 & 0 \end{bmatrix} = TZT' \tag{10}$$

where $\tilde{Z}_{11}$ is $\ell \times \ell$ and $\tilde{Z}_{11} = \tilde{Z}'_{11} > 0$. Letting

$$\tilde{v} = Tv = \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix} \tag{11}$$

with partitioning consistent with Eq. (9) gives

$$v^*Zv = \tilde{v}_1{}^*\tilde{Z}_{11}\tilde{v}_1 \tag{12}$$

and Eq. (8) becomes

$$(Re\lambda + \beta)\tilde{v}_1{}^*\tilde{Z}_1\tilde{v}_1 = 0 \tag{13}$$

If λ is a controllable eigenvalue or an uncontrollable eigenvalue coinciding with one that is controllable, the vector $\tilde{v}_1$ is a nonzero left eigenvector of the controllable part of $T(A+BK)T'$. The definiteness of $\tilde{Z}_{11}$ then gives

$$Re\lambda + \beta = 0 \tag{14}$$

from Eq. (13). For λ an uncontrollable eigenvalue not coinciding with one that is controllable, it can be shown that $\tilde{v}_1 = 0$, which completes the proof.

Theorem 2 establishes that the controllable eigenvalues of $A+BK$ have degree of stability exactly β. Additional properties observed from numerical experience with the extended Bass algorithm and a discrete variable analog of the extended Bass method[4] can be found in Ref. 5.

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